Multifractality of quasihomogeneous states

A. Bershadskii
P.O. Box 39953, Ramat-Aviv 61398, Tel-Aviv, Israel
(Received 26 August 1996)

The hypothesis that *global* multifractality of random systems near a homogeneous state is similar to *local* behavior of the generalized dimension D_q in the vicinity of the capacity dimension D_0 is compared with laboratory experimental data obtained by different authors in low-Reynolds-number flows (at the onset of chaos in the wake of an oscillating cylinder and at near-wall transitional turbulent flow). Competition of two kinds of the multifractality, namely, ''log-normal'' and ''critical,'' is investigated and good agreement with experimental data is established for the case of critical multifractality. [S1063-651X(97)11805-0]

PACS number(s): 47.27.−i, 05.45.+b

If we study some *positive* field χ on some uniform space grid of size r, then we can define the partition function (see, for instance, [1,2])

$$Z_q(r)=\sum_{i=1}^{N}[\chi_i(r)]^q, \tag{1}$$

where $\chi_i(r)$ is defined on the grid in a suitable way and N is the number of the boxes of the grid. If there exists scaling

$$Z_q(r)\sim r^{\tau_q} \tag{2}$$

then the generalized dimension is defined as

$$D_q=\frac{\tau_q}{(q-1)} \tag{3}$$

The Legendre transform

$$\alpha=\frac{d\tau_q}{dq}, \quad f(\alpha)=\alpha q-\tau_q \tag{4}$$

defines the so-called multifractal (or singular) spectrum.

We make the hypothesis that the *global* multifractality (i.e., one valid for a wide range of values of q) of random systems near homogeneous state is similar to the *local* behavior of the generalized dimension D_q in the vicinity of D_0 (where D_0 is the capacity dimension).

Thus we should first study the general behavior of D_q in the vicinity of $q=0$. Let us expand $f(\alpha(q))$ in the Taylor series in this vicinity, i.e.,

$$f(q)=f_0+\left(\frac{df}{dq}\right)_{q=0}q+\frac{1}{2}\left(\frac{d^2f}{dq^2}\right)_{q=0}q^2+\cdots . \tag{5}$$

Since

$$\frac{df}{dq}=q\,\frac{d\alpha}{dq} \tag{6}$$

the *normal* case is

$$\left(\frac{df}{dq}\right)_{q=0}=0. \tag{7}$$

Then the first nontrivial approach to $f(q)$ in this vicinity for the ''normal'' case is

$$f(q)\simeq f_0+\frac{1}{2}\left(\frac{d^2f}{dq^2}\right)_{q=0}q^2. \tag{8}$$

Formally, we can consider also a ''critical'' case with

$$\lim_{q\to+0}\left(\frac{df}{dq}\right)=-a, \tag{9}$$

$$\lim_{q\to-0}\left(\frac{df}{dq}\right)=a, \tag{10}$$

where $\infty>a>0$. For the critical case

$$\alpha=-a\ \ln|q|+C_+ \quad (q>0) \tag{11}$$

and

$$\alpha=a\ \ln|q|+C_- \quad (q<0), \tag{12}$$

where C_+ and C_- are some constants. In the critical case the first nontrivial approach to $f(q)$ is

$$f(q)\simeq f_0-aq \quad (q>0) \tag{13}$$

and

$$f(q)\simeq f_0+aq \quad (q<0). \tag{14}$$

It should be noted that the critical approximation may be nonrealizable in the vicinity of $q=0$. Our hypothesis, however, gives a possibility to observe this kind of multifractality for moderate and large values of q while in the vicinity of $q=0$ itself the normal case is realized (see comparison with experimental data below).

For τ_q we obtain in the normal case

$$\tau_q\simeq-f_0+cq+\frac{1}{2}\left(\frac{d^2f}{dq^2}\right)_{q=0}q^2, \tag{15}$$

where c is some constant which can be found from Eq. (3). It follows from Eq. (3) that $\tau_1=0$ and, consequently, that

$$c = f_0 - \frac{1}{2}\left(\frac{d^2 f}{dq^2}\right)_{q=0}.$$

Then

$$D_q \simeq f_0 + \frac{1}{2}\left(\frac{d^2 f}{dq^2}\right)_{q=0} q \tag{16}$$

[recalling that $(d^2 f/dg^2)_{q=0} < 0$]. It is well known that this representation of D_q is complete for the case of log-normal probability density function of χ (see, for instance, [3–5] and references therein).

On the other hand, for the critical case, one has

$$\tau_q \simeq -aq\ \ln q + (a + C_+)q - f_0 \quad (q>0) \tag{17}$$

and

$$\tau_q \simeq aq\ \ln|q| - (a - C_-)q - f_0 \quad (q<0). \tag{18}$$

From Eq. (3) ($\tau_1 = 0$) we obtain C_+ so that

$$\tau_q \simeq -aq\ \ln q + f_0 q - f_0 \quad (q>0). \tag{19}$$

We now compare these results with experimental data using the relationship [2]

$$\mu_q = d(q-1) - \tau_q, \tag{20}$$

where

$$\langle (\chi_r)^q \rangle \sim r^{-\mu_q}. \tag{21}$$

In particular, we shall consider the case $\chi_r = \epsilon_r$ with

$$\varepsilon_r = \frac{\int_v \varepsilon\ dv}{v},$$

where ε is the turbulent energy dissipation field, $v \sim r^d$ is the volume of a cell of the grid, and d is the topological dimension of the grid.

In the experiments under consideration (see below) $f_0 = d$. If we substitute Eq. (19) into Eq. (20), we obtain for the critical case

$$\mu_q \simeq aq\ \ln q. \tag{22}$$

Figure 1 shows low-Reynolds-number data at the onset of chaos in the wake of an oscillating cylinder [6,2] (the generalized dimensions determined in [6] come from probability densities along a critical attractor at the onset of chaos). One can see good agreement with representation (22).

Figure 2 shows the experimental data obtained in low-Reynolds-number near-wall turbulent flow [by laser Doppler anemometer (LDA) measurements] [7]. And again one can see good agreement with the representation (22). It should be noted that for small values of q the experimental data obtained in the experiment [7] can be fitted by the log-normal multifractality [cf. statement after Eq. (14)]. One can also see from Figs. 1 and 2 that numerical values of a are different in these two cases.

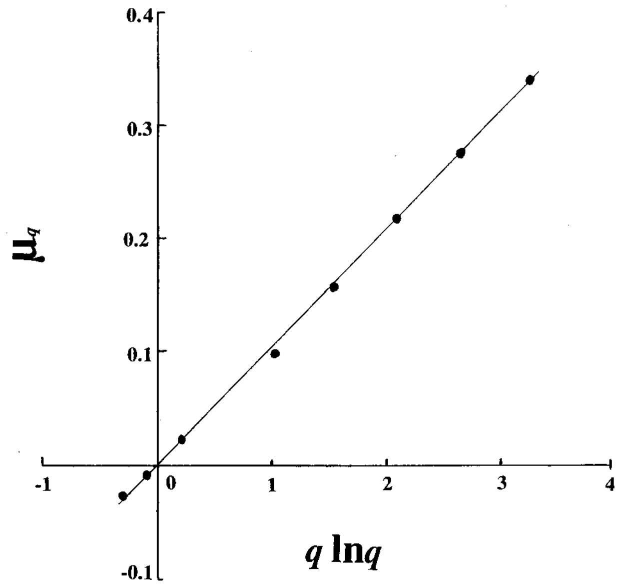

FIG. 1. Experimental low-Reynolds-number data at the onset of chaos in the wake of an oscillating cylinder [6,2]. The solid straight line indicates agreement of these data with Eq. (22).

Finally, let us make some concluding remarks. Two things are happening simultaneously. One of them is a competition between log-normal and critical multifractalities in a general situation. The second is the difference between low- and high-Reynolds-number behaviors. We have shown that low-Reynolds-number data can be fitted by the critical multifractality even though, near $q=0$, log-normal approximation appears valid. On the other hand, high-Reynolds-number data (such as representative in [8,9]) could not be fitted by the critical multifractality approximation (22) directly and an additional consideration should be made in this case. Another interesting question is if, for a general case, the notion of critical has any relevance for large q. One should expect a competition between critical-like multifractality and linear asymptotic if the answer to this question is positive.

The author is grateful to C. H. Gibson and to G. P. Romano, and for information and for stimulating comments and recommendations.

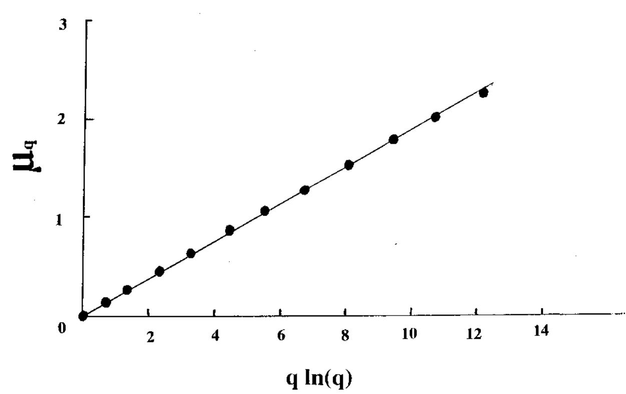

FIG. 2. Experimental data obtained in low-Reynolds-number near-wall turbulent flow (by LDA measurements) [7]. The solid straight line indicates agreement of these data with Eq. (22).

[1] H. E. Stanley and P. Meakin, Nature (London) **335**, 405 (1988).
[2] A. B. Chhabra, R. V. Jensen, and K. R. Sreenivasan, Phys. Rev. A **40**, 4593 (1989).
[3] C. H. Gibson, Proc. R. Soc. London, Ser. A **434**, 149 (1991).
[4] V. R. Chechetkin, V. S. Lutvinov, and Yu. Turgin, J. Stat. Phys. **61**, 573 (1990).
[5] K. R. Sreenivasan and R. R. Prasad, Physica D **38**, 322 (1989).
[6] D. Olinger and K. R. Sreenivasan, Phys. Rev. Lett. **60**, 797 (1987).
[7] G. P. Romano (unpublished).
[8] C. Meneveau and K. R. Sreenivasan, Phys. Rev. Lett. **59**, 797 (1987).
[9] C. Meneveau and K. R. Sreenivasan, J. Fluid Mech. **224**, 429 (1991).